

Mathematics Standards

Third Edition

for teachers of students ages 11–18+

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*National Board Certification
Promotes Better Teaching,
Better Learning, Better Schools*

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Table of Contents

Preface	4
About the National Board for Professional Teaching Standards	4
About the Standards	6
About Certification	7
Foundation of National Board Certification for Teachers	8
Five Core Propositions	8
Architecture of Accomplished Teaching	11
Standards	12
Introduction	12
Mathematics Standards Statements	16
Standard I: Commitment to Mathematics Learning of All Students	18
Knowledge of Mathematics, Students, and Teaching	21
Standard II: Knowledge of Mathematics	21
Standard III: Knowledge of Students	30
Standard IV: Knowledge of the Practice of Teaching	32
The Teaching of Mathematics	36
Standard V: Learning Environment	36
Standard VI: Ways of Thinking Mathematically	38
Standard VII: Assessment	41
Professional Development and Outreach	43
Standard VIII: Reflection and Growth	43
Standard IX: Families and Communities	46
Standard X: Professional Community	48
Standards Committees	50
Acknowledgments	52

Preface

About the National Board for Professional Teaching Standards

The National Board for Professional Teaching Standards (National Board) is a not-for-profit professional organization, created and governed by practicing teachers and their advocates. The founding mission of the National Board is to advance the quality of teaching and learning by

- maintaining high and rigorous standards for what accomplished teachers should know and be able to do;
- providing a national voluntary system certifying teachers who meet these standards; and
- advocating related education reforms to integrate National Board Certification into American education and to capitalize on the expertise of National Board Certified Teachers.

Recognized as the “gold standard” in teacher certification, the National Board believes higher standards for teachers means better learning for students.

Founded in 1987, the National Board began by engaging teachers in the development of standards for accomplished teaching and in the building of an assessment—National Board Certification—that validly and reliably identifies when a teacher meets those standards. Today, there are 25 certificate areas that span 16 content areas and four student developmental levels. The essence of the National Board’s vision of accomplished teaching is captured in the enduring document *What Teachers Should Know and Be Able to Do*, at the heart of which are the Five Core Propositions:

1. Teachers are committed to students and their learning.
2. Teachers know the subjects they teach and how to teach those subjects to students.
3. Teachers are responsible for managing and monitoring student learning.
4. Teachers think systematically about their practice and learn from experience.
5. Teachers are members of learning communities.

The National Board believes that board certification should become the norm, not the exception, and should be fully integrated into the fabric of the teaching profession. In other professions, such as medicine, engineering, and architecture, board certification has helped to create a culture of accomplished practice and is a major reason why those professions are held in such high regard by the public. Those professions did what teaching must now do: strengthen the coherent pipeline of preparation that begins in pre-service and continues through board certification and beyond, with each step engineered to help teachers develop toward accomplished. More than 110,000 teachers had achieved board certification by 2014, a number which represents the largest group of identified teaching experts in the country. Given the size of the teaching workforce, however, this sizable number represents fewer than 3 percent of teachers.

For most children that means they go through their entire schooling without being taught by a board-certified teacher. Each teacher who pursues board certification helps to close this gap, strengthening the profession and the quality of teaching and learning. In a world where board certification is the standard that all teachers aspire to and most achieve, students experience accomplished teaching throughout their schooling, unleashing their potential.

About the Standards

Every child deserves an accomplished teacher—one who is qualified to equip students with the skills to succeed in a global community. The core mission of the National Board for Professional Teaching Standards is to create field-specific standards for accomplished teaching that are grounded in the Five Core Propositions and that articulate the actions that accomplished teachers employ to advance student learning. Each standards document represents a professional consensus on the attributes of practice that distinguish accomplished teaching in that field. Many school systems use the standards as the basis for ongoing professional development, and many colleges and universities incorporate the standards into their undergraduate and graduate teacher education programs.

Standards are developed and revised by a committee of 12–15 members who are representative of accomplished professionals in their field. A majority of standards committee members are practicing Board certified teachers. Other committee members are experts in academic content and child development, including teacher educators, researchers, and other professionals in the relevant field. Standards are disseminated widely for public comment and subsequently revised as necessary before adoption by the National Board's Board of Directors.

Throughout the development of both the standards and the certification process, the National Board ensures broad representation of the diversity that exists within the profession; engages pertinent disciplinary and specialty associations at key points in the process; collaborates closely with appropriate state agencies, academic institutions, and independent research and education organizations; and establishes procedures to detect and eliminate instances of external and internal bias.

National Board Standards and certifications are defined by the developmental level of the students and by the subject or subjects being taught. Teachers select the subject area that makes up the substantive focus of their teaching. They may choose Generalist certificates if they do not focus on one particular subject area in their practice. The four overlapping student developmental levels (listed below) indicate the age of the majority of their students.

- Early Childhood (EC)—ages 3–8
- Middle Childhood (MC)—ages 7–12
- Early Adolescence (EA)—ages 11–15
- Adolescence and Young Adulthood (AYA)—ages 14–18+

About Certification

National Board Certification® is a voluntary, standards-based process designed for teachers to transform the Five Core Propositions into practice. In order to be eligible for certification a teacher must

- Hold a baccalaureate degree from an accredited institution¹;
- Have a minimum of three years' teaching experience at the early childhood, elementary, middle school, or high school level; and
- Where it is required, hold a state teaching license.

The assessments, aligned with the Five Core Propositions and the standards, are designed so that teachers demonstrate their practice by providing evidence of what they know and do. The evidence-based assessment honors the complexities and demands of teaching.

In 2014, the National Board initiated revision of the assessment to make the process more flexible, affordable, and efficient for teachers. In all certificate areas, candidates for National Board Certification are now required to complete four components: three portfolio entries, which are submitted online, and a computer-based assessment, which is administered at a testing center. Teachers develop portfolio entries that require analysis of their practice as it relates to student learning and to being a reflective, effective practitioner. Designed to capture what a teacher knows and is able to do in real time and in real-life settings, the portfolio consists of description, analysis, and reflection focused on student learning that is captured on video and in student work samples. The process requires teachers to reflect on the underlying assumptions of their practice and the impacts of that practice on student learning.

Teachers also demonstrate content knowledge by responding to open-ended and multiple choice questions delivered at a secure testing site. The assessment center component complements the portfolio, validates that the knowledge and skills exhibited in the portfolio are accurate reflections of what a candidate knows, and provides candidates with opportunities to demonstrate knowledge and skills not sampled in the portfolio.

Assessments are based on the standards and are developed for every certificate area by educators who specialize in the same content and student developmental level as the candidates. Educators who are themselves practitioners in the certificate area score the submitted portfolio entries. They must successfully complete intensive training and qualify for scoring on the basis of their understanding of National Board Standards and scoring guidelines.

¹ Candidates registering for the Career and Technical Education certificate are required to hold a bachelor's degree only if their state required one for their current license.

Foundation of National Board Certification for Teachers

Five Core Propositions

The National Board framework for accomplished teaching was established in its 1989 publication, *What Teachers Should Know and Be Able to Do*. The Five Core Propositions serve as the foundation for all National Board standards and assessments, defining the level of knowledge, skills, abilities, and commitments that accomplished teachers demonstrate. Teachers embody all Five Core Propositions in their practices, drawing on various combinations of these skills, applications, and dispositions to promote student learning.

1. Teachers are committed to students and their learning.

Accomplished teachers base their practice on the fundamental belief that all students can learn and meet high expectations. They treat students equitably, recognizing the individual differences that distinguish one student from another and taking account of these differences in their practice. They adjust their practice based on observation and understanding of their students' interests, abilities, skills, knowledge, language, family circumstances, and peer relationships. They view students' varied backgrounds as diversity that enriches the learning environment for every student.

Accomplished teachers understand how students develop and learn. They consult and incorporate a variety of learning and development theories into their practice, while remaining attuned to their students' individual contexts, cultures, abilities, and circumstances. They are committed to students' cognitive development as well as to students' ownership of their learning. Equally important, they foster students' self-esteem, motivation, character, perseverance, civic responsibility, intellectual risk taking, and respect for others.

2. Teachers know the subjects they teach and how to teach those subjects to students.

Accomplished teachers have a rich understanding of the subject(s) they teach and appreciate how knowledge in their subject is created, organized, linked to other disciplines, and applied to real-world settings. While maintaining the integrity of disciplinary methods, content, and structures of organization, accomplished teachers develop the critical and analytical capacities of their students so they can think for themselves.

Accomplished teachers command specialized knowledge of how to convey and reveal subject matter to students. They are aware of the preconceptions and background knowledge that students typically bring to each subject and draw upon pedagogical and subject matter understandings to anticipate challenges,

modify their practice, and respond to students' needs. They also demonstrate a commitment towards learning about new strategies, instructional resources, and technology that can be of assistance. Their instructional repertoire and professional judgment allow them to generate multiple paths to knowledge in the subjects they teach, and they are adept at teaching students how to pose and solve their own problems so they can continue exploring and advancing their understanding.

3. Teachers are responsible for managing and monitoring student learning.

Accomplished teachers view themselves as facilitators of student learning within dynamic instructional settings. They create, enrich, maintain, and alter learning environments while establishing effective ways to monitor and manage those environments and the student learning that occurs within them. They possess a comprehensive knowledge of instructional methods, know when each is appropriate, and can implement them as needed. They use instructional time constructively and efficiently, customizing physical layout, resources, and instructional methods. They enlist the knowledge and support of a wide range of stakeholders to provide their students with enriched opportunities to learn. They understand the strengths and weaknesses of pedagogical approaches they may take, as well as the suitability of these approaches for particular students.

Accomplished teachers know how to engage students in varied settings and group configurations. They create positive and safe learning environments that guide student behavior and support learning, allowing the schools' goals for students to be met. They are adept at setting norms for social interaction among students and between students and teachers. They understand how to motivate students and value student engagement, supporting them as they face and learn from challenges.

Accomplished teachers assess the progress of individual students as well as that of the class as a whole. They apply their knowledge of assessment to employ multiple methods for measuring student growth and understanding. They use the information they gather from monitoring student learning to inform their practice, and they provide constructive feedback to students and families. They collaborate with students throughout the learning process and help students engage in self-assessment.

4. Teachers think systematically about their practice and learn from experience.

Accomplished teachers possess a professional obligation to become perpetual students of their craft. Committed to reflective learning, they are models of educated persons. They exemplify the virtues they seek to inspire in students—curiosity, honesty, fairness, respect for diversity and appreciation of cultural differences—and the capacities that are prerequisites for intellectual growth: the ability to reason and take multiple perspectives, to be creative and take risks, and to adopt an experimental and problem-solving orientation.

Accomplished teachers draw on their knowledge of human development, subject matter, and instruction, and their understanding of their students to make principled judgments about sound practice. Their decisions are not only grounded in established theories, but also in reason born of experience. They engage in lifelong learning, which they seek to encourage in their students.

Accomplished teachers seek opportunities to cultivate their learning. Striving to strengthen their teaching and positively impact student learning, teachers use feedback and research to critically examine

their practice, seek to expand their repertoire, deepen their knowledge, sharpen their judgment and adapt their teaching to new findings, ideas and theories.

5. Teachers are members of learning communities.

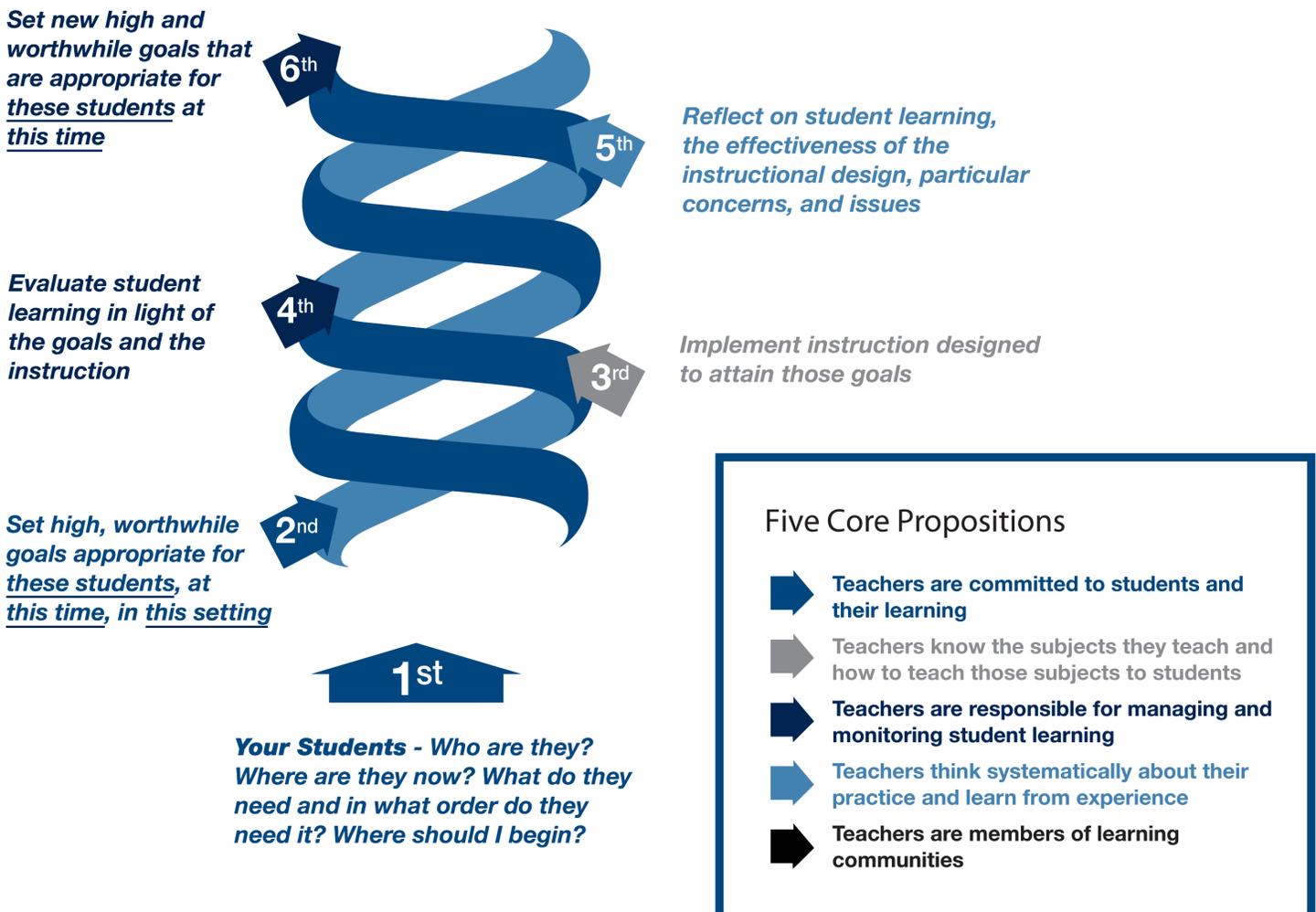
Accomplished teachers participate actively in their learning communities to promote progress and achievement. They contribute to the effectiveness of the school by working collaboratively with other professionals on policy decisions, curriculum development, professional learning, school instructional programs, and other functions that are fundamental to the development of highly productive learning communities. They work collaboratively and creatively with families and the community, engaging them productively in the work of the school and cultivating students' connections with the opportunities, resources, and diversity they afford.

Accomplished teachers can evaluate school progress and the allocation of school resources in light of their understanding of state and local educational objectives and their knowledge of student needs. They are knowledgeable about and can advocate for specialized school and community resources that can be engaged for their students' benefit, and are skilled at employing such resources as needed.

Architecture of Accomplished Teaching

The Architecture of Accomplished Teaching provides a view of how the use of the Five Core Propositions and the standards that are developed from them result in student learning. As depicted in the Architecture of Accomplished Teaching illustration, shown below, one strand represents teaching practice as grounded in the Five Core Propositions, while the other strand represents the teacher’s impact on students and their learning.

The Architecture of Accomplished Teaching: *What is underneath the surface?*



The National Board program certifies accomplished teachers who positively influence student learning through effective teaching practice. The process includes the core propositions for all teachers, a common set of accomplished teaching standards specific to the content field and students’ developmental levels, and a set of evidence-based assessments specific to the field that certify what accomplished teachers know and do.

Standards

Introduction

Balancing precision, rigor, and reasoning with opportunities for students to marvel at the elegance and often surprise embedded in doing mathematics, accomplished mathematics teachers help students understand the field as a place of truth, offering insight into the persistent human effort to make sense of the world's order, chaos, stability, and change. In a world where unexamined assumptions are often held as true, mathematicians pursue questions of “why” or “how,” demonstrating a remarkable history of intellectual service to problem solving and decision making across time and cultures. Accomplished mathematics teachers seek to support students' learning of not only procedures and concepts, but also to appreciate the wonder and beauty of mathematical problems, solutions, and connections that draw students into further study and call professional mathematicians into the field.

Accomplished mathematics teachers help students grapple with fundamental concepts such as quantity, space, change, and chance, critical to understanding both mathematics and the myriad of disciplines that rely on mathematical ideas. While many believe mathematics is the sum of a variety of procedures, accomplished teachers know these procedures are connected to fundamental underlying concepts. In the classrooms of accomplished teachers, students are engaged in identifying patterns; solving problems; reasoning; forming and testing conjectures, justification, and proof; and communicating results. Students search for connections and solve problems, while reflecting on both the mathematics and their own thought processes.

Accomplished mathematics teachers appreciate the richly interconnected nature of the discipline and share that with students. Teachers identify tasks for students that communicate connections between mathematics topics and between mathematics and the world. They not only choose tasks related to everyday life—to the sciences, to economics, to politics, or to business—but they also choose tasks that will extend understanding within mathematics. Their choice of problem contexts reflects the breadth of mathematics and its applications.

Accomplished mathematics teachers use their knowledge of mathematics and of how students learn to create a stimulating and productive environment in which students are empowered to do mathematics. Teachers realize that teaching students to “think mathematically” means helping them develop a mathematical point of view in which they consistently use mathematical thinking processes; recognize situations in which mathematical reasoning might be useful; and have the abilities, skills, and confidence to take action. To encourage mathematical thinking, teachers provide opportunities for students to test their mathematical ideas, applying their growing knowledge to a variety of problems.

Accomplished teachers provide multiple opportunities to teach students to use mathematics in new and meaningful ways. Teachers help students acquire confidence in learning, doing, and understanding mathematics, becoming critical consumers who use mathematics to evaluate, analyze, and synthesize

information. Mathematics teachers have an “eye to the future,” knowing and communicating how the mathematics learned now will relate to students’ future work or education. The global nature of today’s workforce requires students who are mathematically literate and able to contribute solutions to real-world problems.

Accomplished mathematics teachers consistently use clear and unambiguous language when communicating mathematics. They have a command of vocabulary, symbols, notation and concepts required to make sense of mathematics. Accomplished teachers deliberately structure opportunities for students to use and develop appropriate mathematical discourse as they reason and solve problems. These teachers give students opportunities to talk with one another, to work together in solving problems, and to use both written and oral discourse to describe and discuss their mathematical thinking and understanding. As students talk and write about mathematics—as they explain their thinking—they deepen their mathematical understanding in powerful ways.

Teaching mathematics is invigorating, as demonstrated by the accomplished teacher’s use of different methods and strategies to engage students in the beauty and exhilaration of mathematics. Because learning mathematics is sometimes challenging, teaching it is rewarding and significant—especially when a disenchanting student does well, enhancing their self worth and making their potential seems as limitless to them as to the teacher.

Accomplished teachers value mathematics. They take joy in mathematics and communicate that joy to students. The accomplished teacher appreciates how knowledge in mathematics is created and uses the power of mathematics to fascinate students, providing them opportunities to experience the intellectual satisfaction that comes from finding a solution to a problem or justifying a conjecture.

Recognizing the important and dynamic nature of mathematics, several changes have been made by the standards committee to the *Mathematics Standards* of the National Board for Professional Teaching Standards. The most significant change is the merging of mathematics standards previously published as separate documents for two developmental levels, Early Adolescence and Adolescence and Young Adulthood. In reviewing the previous mathematics standards, the committee concluded that accomplished mathematics pedagogy is the same regardless of student developmental level, with differences surfacing in the core mathematical knowledge expected of the Early Adolescence (EA) teacher versus the Adolescence and Young Adulthood (AYA) teacher. The committee spent considerable time distinguishing what EA and AYA teachers of mathematics should know, resulting in the disclaimer at the beginning of the “Core Mathematical Knowledge” section of Standard II, “Accomplished teachers understand the major ideas in the core domains of mathematics. Although their expertise may vary in degree for particular domains, teachers have a fundamental knowledge base from which to build student mathematical understanding. However, within certain domains there are differences between the application of knowledge by the EA teacher and the AYA teacher. When this distinction occurs the AYA teacher is specifically referenced.” The committee recognized that high school teachers need to know concepts taught in middle school and both EA and AYA teachers must be familiar with mathematics at the next level, thus a merging of the EA and AYA standards contributes to the spectrum of what is expected of accomplished mathematics teachers.

There is a growing national consensus that the level of mathematical knowledge needed to graduate a productive citizenry is increasing. Thus, other issues driving changes in the mathematical knowledge standard include trends to teach core algebraic concepts in middle school and to require higher levels of mathematics for high school graduation. To reflect these higher standards for students, the committee

increased the rigor and precision of the core mathematical knowledge for the accomplished EA and AYA teacher over that found in the previous mathematics standards. Finally, the committee recognized the mathematics community is moving toward a cohesiveness of practice, collaboration, cooperation, and communication that informed the merging of EA and AYA mathematics standards.

In addition to merging the two mathematics standards documents, the committee combined several standards that were separated in one or both of the previous documents. One change is consolidation of the standard on diversity, fairness, and equity with the standard on commitment to mathematics learning. This change is a result of the committee's belief that "all students" is really "all students." Because the concepts of the art and knowledge of teaching are interdependent, the previously separated standards were combined to create one standard, "Knowledge of the Practice of Teaching."

In addition to merging the two documents and some standards the committee used more precise language in this new edition. For example, in Standards IX and X, "community" has been delineated clearly among the family community, the school community, and the professional community. As well, several terms are used with specific, yet often varying, definitions throughout the document. For example, curriculum can be defined in multiple ways; in this document unless otherwise noted, curriculum means the mathematics that is taught in the classroom. At times, however, and noted accordingly, curriculum refers to written standards. Finally, when referring to language skills, the committee recognized the barriers that language often can present. To that end, the committee provided pathways for English language learners to acquire mathematical knowledge and skills while also learning the academic language of mathematics.

These standards reflect an ideal that accomplished teachers strive to obtain, forever growing in their knowledge and practice in order to empower their students to learn and to use mathematics. The accomplished teacher's passion for mathematics and for teaching mathematics is evident in their efforts to grow professionally and in their work with students. Accomplished teachers of mathematics challenge themselves and challenge students to use mathematics in consistent, expected ways as well as in new, creative ways. Thinking mathematically includes representing, modeling, proving, experimenting, conjecturing, classifying, visualizing, and computing—all ways in which to approach mathematics and life.

Developing High and Rigorous Standards for Accomplished Practice

Mathematics Standards describes what accomplished teachers should know and be able to do. The standards are meant to reflect the professional consensus at this point about the essential aspects of accomplished practice. The deliberations of the Mathematics Standards Committee were informed by various national and state initiatives on student and teacher standards that have been operating concurrently with the development of NBPTS Standards. As the understanding of teaching and learning continues to evolve over the next several years, these standards will be updated again.

An essential tension of describing accomplished practice concerns the difference between the analysis and the practice of teaching. The former tends to fragment the profession into any number of discrete duties, such as designing learning activities, providing quality explanation, modeling, managing the classroom, and monitoring student progress. Teaching as it actually occurs, on the other hand, is a seamless activity.

Everything an accomplished teacher knows through study, research, and experience is brought to bear daily in the classroom through innumerable decisions that shape learning. Teaching frequently requires

balancing the demands of several important educational goals. It depends on accurate observations of particular students and settings, and it is subject to revision on the basis of continuing developments in the classroom.

The paradox, then, is that any attempt to write standards that dissect what accomplished teachers know and are able to do will, to a certain extent, misrepresent the holistic nature of how teaching actually takes place. Nevertheless, the fact remains: Certain identifiable commonalities characterize the accomplished practice of teachers. The standards that follow are designed to capture the knowledge, artistry, proficiency, and understandings—both deep and broad—that contribute to the complex work that is accomplished teaching.

The Standards Format

Accomplished teaching appears in many different forms, and it should be acknowledged at the outset that these specific standards are not the only way it could have been described. No linearity, atomization, or hierarchy is implied in this vision of accomplished teaching, nor is each standard of equal weight. Rather, the standards are presented as aspects of teaching that are analytically separable for the purposes of this standards document but that are not discrete when they appear in practice.

The report follows a two-part format for each of the standards:

- **Standard Statement**—This is a succinct statement of one vital aspect of the practice of the accomplished teacher of mathematics. Each standard is expressed in terms of observable teacher actions that have an impact on students.
- **Elaboration**—This passage provides a context for the standard, along with an explanation of what teachers need to know, value, and do if they are to fulfill the standard. The elaboration includes descriptions of teacher dispositions toward students, their distinctive roles and responsibilities, and their stances on a range of ethical and intellectual issues that regularly confront them.

In addition, throughout the document are examples illustrating accomplished practice and demonstrating how decisions integrate various individual considerations and cut across the standard document. If the standards pull apart accomplished teaching into discrete elements, the examples put them back together in ways more clearly recognizable to teachers. Because the National Board believes there is no single “right” way to teach students, these examples are meant to encourage teachers to demonstrate their own best practice.

Mathematics Standards Statements

The National Board for Professional Teaching Standards has organized the standards for accomplished teachers of mathematics into the following ten standards. The standards have been ordered to facilitate understanding, not to assign priorities. They each describe an important facet of accomplished teaching; they often occur concurrently because of the seamless quality of accomplished practice. These standards serve as the basis for National Board Certification in this field.

Standard I: Commitment to Mathematics Learning of All Students

Accomplished mathematics teachers acknowledge and value the individuality and worth of each student, believe that every student can learn and use mathematics, and are dedicated to their success. Accomplished mathematics teachers are committed to the fair and equitable treatment of all students—especially in their learning of mathematics.

Knowledge of Mathematics, Students, and Teaching

Standard II: Knowledge of Mathematics

Accomplished mathematics teachers have a deep and broad knowledge of the concepts, principles, techniques, and reasoning methods of mathematics, and they use this knowledge to inform curricular goals and shape their instruction and assessment. They understand significant connections among mathematical ideas and the applications of these ideas to problem solving in mathematics, in other disciplines, and in the world outside of school.

Standard III: Knowledge of Students

Accomplished teachers use their knowledge of human development and individual students to guide their planning and instructional decisions. They understand the impact of prior mathematical knowledge, home life, cultural background, individual learning differences, student attitudes and aspirations, and community expectations and values on students and their mathematics learning.

Standard IV: Knowledge of the Practice of Teaching

Accomplished mathematics teachers use their knowledge of pedagogy along with their knowledge of mathematics and student learning to inform curricular decisions; select, design, and develop instructional strategies and assessment plans; and choose materials and resources for mathematics instruction. Accomplished mathematics teachers stimulate and facilitate student learning by using a wide range of practices.

The Teaching of Mathematics

Standard V: Learning Environment

Accomplished mathematics teachers create environments in which students are active learners, show willingness to take intellectual risks, develop self-confidence, and value mathematics. This environment fosters student learning of mathematics.

Standard VI: Ways of Thinking Mathematically

Accomplished mathematics teachers develop their own and their students' abilities to reason and think mathematically—to investigate and explore patterns, to discover structures and establish mathematical relationships, to formulate and solve problems, to justify and communicate conclusions, and to question and extend those conclusions.

Standard VII: Assessment

Accomplished mathematics teachers integrate a range of assessment methods into their instruction to promote the learning of all students by designing, selecting, and ethically employing assessments that align with educational goals. They provide opportunities for students to reflect on their strengths and weaknesses in order to revise, support, and extend their individual performance.

Professional Development and Outreach

Standard VIII: Reflection and Growth

To improve practice, accomplished mathematics teachers regularly reflect on what they teach, how they teach, and how their teaching impacts student learning. They keep abreast of changes and learn new mathematics and mathematical pedagogy, continually improving their knowledge and practice.

Standard IX: Families and Communities

Accomplished mathematics teachers collaborate with families and communities to support student engagement in learning mathematics. They help various communities, within and outside the school building, understand the role of mathematics and mathematics instruction in today's world.

Standard X: Professional Community

Accomplished mathematics teachers continually collaborate with other teachers and education professionals to strengthen the school's mathematics program, promote program quality and continuity across grade levels and courses, and improve knowledge and practice in the field of mathematics education.

Standard I

Commitment to Mathematics Learning of All Students

Accomplished mathematics teachers acknowledge and value the individuality and worth of each student, believe that every student can learn and use mathematics, and are dedicated to their success. Accomplished mathematics teachers are committed to the fair and equitable treatment of all students—especially in their learning of mathematics.

Commitment to Diverse Learners

Accomplished teachers base their decisions about the teaching of mathematics on the belief that all students can learn. Teachers¹ continually determine each student's level of mathematical knowledge and understanding and build on that foundation. They are alert and sensitive to the diversity that exists in students' prior learning experiences; individual learning approaches; family,² cultural, and economic backgrounds; students' interests; and their special needs. Teachers recognize the beliefs and attitudes toward mathematics that each student brings to the classroom and promote a respect for the value of mathematics. Teachers are aware that any of these factors, as well as others, can affect how students approach the learning of mathematics. Strategies for engaging all students may come from current research, collaboration, personal experience, and professional development.

Accomplished teachers are dedicated to meeting the needs of a diverse student population. Teachers confront issues of diversity proactively to promote academic and social equity, maintaining high expectations for all learners. Mathematics teachers actively and positively challenge their own and others' biased behaviors and stereotypical perspectives. For instance, a teacher might examine why one gender is significantly outperforming the other in algebra classes. Teachers are keenly aware of the historical perspectives and biases that have created social and academic barriers for students and work to remove these obstacles, such as less rigorous mathematics for students in lower-level courses. Teachers ensure that their students receive equitable opportunities to learn and advance in mathematics by maintaining

¹ All references to *teachers* in this document, whether stated explicitly or not, refer to accomplished teachers of mathematics.

² *Family* is used in this document to refer to the people who are the primary caregivers, guardians, and significant adults of children.

the focus on standards-based concepts and skills, and they act to dispel the notion that not all students are capable of learning mathematics.

Accomplished teachers value the importance of their students' diverse cultures and backgrounds. Teachers build on the richness of the heritage and culture of all their students and give students opportunities to think in ways that are both culturally familiar and unfamiliar. Teachers recognize the unique contributions and perspectives each student brings to the learning environment. For example, the teacher may bring artwork representing the cultures of students in the school, such as Native American pottery, African American quilts, or Middle Eastern mosaics, to discuss topics of geometry such as transformations and tessellations. Teachers use this knowledge to foster positive interaction in the classroom and to support each student's mathematical growth. Teachers are aware that students' cultural backgrounds and life experiences can influence the ways they interact in the classroom and the ways students approach and learn mathematics.

Accomplished teachers are aware of the supportive attention that must be given to students who are learning English as a new language. Teachers ensure that such students are able to understand instruction and participate in class and small-group discussions; teachers may also give students who are learning English as a new language alternative assignments and assessments so that their ability to demonstrate understanding and proficiency in mathematics does not depend on their proficiency in English. Teachers work to ensure that such accommodations are made so that all students have equitable access to appropriate learning opportunities.

Accomplished teachers are aware of the issues involved in providing instruction to students with exceptionalities, including students with gifts and talents. Teachers modify curriculum, instruction, and assessments as necessary. They comply with federal, state, and local laws, regulations, and policies concerning students with unique needs. Teachers work closely with the specialists and support personnel who have valuable insights into these students, and teachers willingly team with these personnel to ensure that these students have every opportunity to achieve their educational goals and objectives. Teachers advocate for and, when possible, make use of assistive technologies—for instance, computers with voice-recognition or speech-synthesis software that can enable students with exceptional needs to communicate their thought processes and mathematical arguments.

Commitment to Mathematics Learning

Accomplished teachers help students acquire confidence in learning, doing, and understanding mathematics. Mathematics teachers focus on students, their activities, and their mathematical proficiency. In order to be mathematically proficient, students need to be able to understand the underlying concepts, achieve fluency and accuracy with procedures and algorithms, use several strategies to solve problems, communicate their thinking, understand the value of the mathematics, and believe in their ability to learn it. Teachers make the phrase “mathematics for all” come alive in their classrooms. They strive to inspire students to work diligently

to learn mathematics and encourage them to prioritize making time for learning mathematics. Genuinely committed to students, teachers let students know that they find doing and teaching mathematics a lively and enjoyable experience. With that in mind, a teacher might use gingerbread houses to engage students in topics involving measurement and proportion. Mathematics teachers create opportunities for each student to experience the satisfaction of success.

Accomplished teachers know that mathematical proficiency is essential for everyone and work to encourage all students to take more mathematics courses. Teachers also work to provide opportunities for extra-curricular activities such as mathematics clubs and competitions. Teachers develop special pedagogical strategies for students who come to them with insufficient mathematical preparation in order to bring these students' learning up to course level as quickly as possible. Teachers recognize and work to overcome barriers that might prevent students from succeeding in mathematics. Teachers provide support and encouragement to and establish relationships with families and school personnel to ensure student proficiency in mathematics.

Accomplished teachers take the extra steps required to ensure that students learn and encourage students to advance in mathematics as far as possible. It is important for teachers to know and communicate to students what is expected at the next level of mathematics. Teachers communicate connections among mathematics topics and between mathematics and the world. For example, a teacher could use a system of linear equations to model the total cost of two cell phone plans and use equations and inequalities to discuss with students appropriate domain and range values derived from the real world context. The students could then determine values for which plan would cost less. While teaching geometry topics, another real world context might include an exploration of local architecture and construction projects. Teachers have an "eye to the future," knowing and communicating how the content that mathematics students are learning now will relate to their future work or education.

Knowledge of Mathematics, Students, and Teaching

Accomplished teachers offer all students the opportunity to learn. Only by having a deep and broad understanding of mathematics can teachers organize and deliver instruction that helps students build their own deep and broad understanding of mathematics. Only by knowing their students well can teachers consistently make instructional decisions that will further students' learning. Further, only by skillfully combining their knowledge of students and mathematics with their knowledge about how to teach mathematics can teachers enable students to learn mathematics successfully. The following three standards form the foundation for the decisions and actions taken by accomplished mathematics teachers. They are the basis for the six remaining standards.

Standard II Knowledge of Mathematics

Accomplished mathematics teachers have a deep and broad knowledge of the concepts, principles, techniques, and reasoning methods of mathematics, and they use this knowledge to inform curricular goals and shape their instruction and assessment. They understand significant connections among mathematical ideas and the applications of these ideas to problem solving in mathematics, in other disciplines, and in the world outside of school.

Mathematics is a fundamental tool in the persistent human effort to make sense of the world—its order, chaos, stability, and change. It has applications, for example, in scientific, technological, economic, and political arenas. Although it is one of the oldest disciplines of human knowledge and thought, the field of mathematics continues to grow and evolve. New concepts, principles, and methods become a part of the discipline each year. For example, the concept of fractals and the theory of computational complexity have been developed during the lifetime of many of today's teachers.

Accomplished teachers have a deep and broad understanding of the mathematics well beyond the level they teach. For example, an early adolescence teacher could

demonstrate how transformations of shapes that are studied at the pre-algebra level will relate to transformations of functions encountered in higher-level algebra. Their knowledge encompasses not only the details, rules, and procedures of mathematics, but also the larger themes and connecting ideas that tie together its various strands. In some higher-level mathematics courses, an adolescence young adulthood teacher could guide students to apply their prior knowledge about solutions to linear equations and to systems of linear equations in their study of non-linear equations and systems of non-linear equations. This rich, conceptual knowledge of mathematics allows them to make decisions about what to emphasize in the planning of lessons. Their knowledge base makes them well aware of where their students are headed—individually and as a group—and how to move them to continually deepening levels of mathematical understanding. This knowledge of mathematical principles, ideas, and reasoning allows teachers to monitor and adjust their teaching continuously, directing students toward key understandings that arise naturally from students' work by asking questions and guiding discourse toward these understandings.

To teach effectively, accomplished teachers have a sound foundation in the disciplines that compose mathematics. They understand the history of mathematics and how knowledge in this area has developed over time. They know the ways of thinking, talking, and writing about mathematics and have enough experience with them to share them with their students. They help students develop the ability to think mathematically and to communicate correctly about mathematics both verbally and in writing. These teachers know their field well enough to understand the challenges associated with establishing the body of knowledge that constitutes the field.

Accomplished teachers view the discipline from several perspectives and have a broad and rich understanding of the knowledge base that informs the mathematics curriculum—in particular, number and operation, algebra and functions, geometry, trigonometry, discrete mathematics, data analysis and statistics, and calculus. Teachers are fluent in the skills and conceptual developments within each discipline and draw on this knowledge to design lessons that are both mathematically substantive and pedagogically sound. Teachers are also aware of the role their knowledge of mathematics can play in advancing student learning. Consequently, these teachers demonstrate breadth as well as depth of knowledge to support their teaching.

To make classroom decisions that support student learning, accomplished teachers must understand both mathematics and students and, as teachers, must continue to grow in their understanding. (See [Standard I—Commitment to Mathematics Learning of All Students](#)) for a definition of mathematical understanding. To help students acquire and then build on the ideas, methods, and skills that underlie mathematics; to see relationships among these elements; and to make significant applications of them, mathematics teachers must have a broad and well-integrated knowledge of these underlying ideas, as well as the methods and techniques of mathematics. Teachers must appreciate the richly interconnected nature of the discipline and share that with students.

Accomplished teachers know the productive connections between mathematics and other fields of human endeavor—connections that have given mathematics a remarkable history of intellectual service to problem solving and decision making across time and cultures. Teachers have a broad understanding of the methodology of the axiomatic system and know that this methodology is shared by the physical sciences. Teachers see that mathematics and the sciences in general both rely on recognizing patterns in order to make generalizations and develop understanding. Teachers understand the importance of proof and how it works in establishing truth and in providing a standard of rigor that sets mathematics apart from other disciplines.

A strong force in the contemporary evolution of mathematics—and of mathematics teaching—is the power of modern computational technology. As a result of the growing use of this technology, some problems and topics are becoming more accessible to students, along with new ways to represent and manipulate mathematical information. Accomplished teachers have knowledge of current technology and are fluent with its use.

Contexts for Mathematics

Accomplished teachers understand the foundations of abstract concepts and techniques related to concrete cases, and they use this understanding to make curricular and instructional decisions and to help students make connections across disciplines. Teachers appreciate the historical course through which mathematical ideas have developed and the ways different cultures have influenced and contributed to that development.

An accomplished teacher's knowledge of the context within which mathematics has evolved is useful and includes the following:

- Knowledge of the major threads in the historical development of key mathematical ideas—the conceptual stumbling blocks and insights that provided important breakthroughs—and the contributions of various individuals and cultures to those developments.
- Knowledge of the ways mathematical ideas have been and remain fundamental to practical and scientific progress in fields related to mathematics. This includes applications for the major concepts and techniques of core content topics in the school curriculum, as well as the modeling processes that are fundamental to effective applications of mathematics. Such applications provide a basis for thinking about and using mathematics. Effective use of technology is an essential part of this modeling and application process.
- Knowledge of a set of analytical techniques and the ability to recognize when the techniques are appropriate to apply in real situations.

Mathematics is often described by naming important concepts, facts, and operations in its major topic strands. However, throughout those strands, accomplished mathematics teachers always keep in mind and apply the following essential guiding principles:

- Communication of mathematics with precision—teachers know that mathematics does not tolerate ambiguities and that no mathematical statement can be regarded as correct if it is correct in some, but not all, possible circumstances. For example, the statement “if x, y, z are numbers and $x < y$, then $zx < zy$ ” would be correct to most people because “numbers” to them means only positive numbers, but it must be regarded as an incorrect statement because it is wrong when z is zero or negative.
- Acknowledgement of the need for precise definitions—teachers know that definitions are the bedrock upon which logical arguments rest. They know that an explanation of a mathematical assertion cannot be based on concepts only vaguely understood. For example, without a precise definition of “similarity,” it is impossible to give an explanation of the fact that the graphs of all quadratic functions are similar to each other.
- Ability to support mathematical assertions with reasoning—teachers know that, in mathematics, the truth of a statement is established not by the authority of the person who makes it but by a logical explanation. Teachers can use reasoning to derive standard formulas and algorithms (e.g., the quadratic formula, the area formula of a circle).
- Ability to think about mathematics, not as a collection of isolated facts but as a whole fabric—teachers know (1) the importance of specific mathematics topics (e.g., Why study non-linear functions?); (2) the usefulness of mathematics topics in relation to others (e.g., the criteria for similar triangles are useful for showing that the graph of a linear equation is a straight line); and (3) the connections among mathematics topics (e.g., the concept of division for whole numbers is the same as that of division for fractions, rational numbers, and complex numbers; or the fundamental role played by the concept of congruence in the definition of area or volume).
- Knowledge of mathematics as the focused response to the need for solutions to major problems—for example, geometric measurements arose from the need to compare sizes in farming and bartering, and calculus resulted from the need to study change. Consequently, teachers come to value the central importance of problem solving in mathematics.
- Knowing that solutions to problems are usually not achieved immediately or without considerable thought—the process of solving problems often evolves from simple, heuristic arguments and strategies to precise and rigorous ones. Teachers, therefore, know that part of problem solving includes the use of heuristic arguments as a guide (e.g., the testing of extreme cases, the search for specific examples to shed light on the general case, and the use of visual representations).

Core Mathematical Knowledge

Accomplished teachers understand the major ideas in the core domains of mathematics. Although their expertise may vary in degree for particular domains, teachers have a fundamental knowledge base from which to build student mathematical understanding. However, within certain domains there are differences between the application of knowledge by the early adolescence teacher and the adolescence-young adulthood teacher. In this document, when this distinction occurs, the adolescence-young adulthood teacher is specifically referenced. Core mathematical knowledge includes numbers and operations, algebra and functions, geometry, discrete mathematics, trigonometry, data analysis and statistics, and calculus. (This list of topics is not intended to indicate a specific order of study.)

Numbers and Operations

Accomplished teachers understand basic concepts of numbers and operations and can model them in a variety of ways. Teachers understand the conceptual basis for the number systems—the relationships between and among whole numbers, integers, rational, real, and complex numbers. Teachers understand that the backbone of school mathematics is the rational number system. They should be able to explain the reasoning behind the algorithms of rational number operations. They recognize the pervasiveness of proportionality across mathematical strands and can use that concept as a model in describing a variety of situations, including those calling for ratios and percent. They understand the need for making estimates and also know what situations call for estimates rather than exact answers, and vice versa.

Algebra and Functions

Accomplished teachers recognize algebra as a language for expressing generality and abstraction. They also recognize that a foundational skill in algebra is the correct and fluent use of symbols and expressions containing symbols. They can demonstrate and apply the critical role and ubiquitous nature of the abstract laws of associativity, commutativity, and distributivity in symbolic manipulations, especially in the solution of equations. Accomplished teachers of adolescents and young adults, having internalized the processes of abstraction and generalization and being familiar with the concept of a ring, will recognize the structural similarity between integers and polynomials. Familiar with the concept of a field, they will also recognize the connection between fractions and rational expressions.

In the context of linear functions, accomplished teachers understand the role of units in problems of rate. In that same context they have knowledge of linear transformations and matrices. Teachers know that the quadratic formula is the apex of a development that yields all desirable information about quadratic equations. They know that the definition of an exponential function ax depends on a correct definition of the rational powers of positive numbers and why all such exponential functions with $a > 0$ and $a \neq 1$ are either increasing or decreasing. They know when a function has an inverse function. Mathematics teachers know that the logarithmic function is important in mathematics because it transforms multiplication into addition. They can

use the technique of mathematical induction to write proofs, such as the summation formula of an arithmetic sequence and the binomial theorem. Accomplished teachers also know that algebra deals only with finite processes and that infinite processes, such as limits, belong to calculus. Teachers also understand, however, that algebra can be used to do some things that typically use calculus, such as finding the maximum or minimum of a quadratic function. It is a simple task in calculus, but accomplished teachers know that it can also be done easily using only algebra.

Accomplished teachers understand the interplay among numerical, symbolic, verbal, and graphical representations of quantitative relationships and the role and means of transforming and simplifying these representations. Teachers are proficient in using concepts and symbolic expressions for working with families of functions, such as polynomial, exponential, rational, logarithmic, and trigonometric families.

Geometry

Accomplished teachers know that geometry is the analytic study of spatial information and that the description of spatial information requires precision. For this reason, exact definitions are extremely important, and accomplished teachers are fluent in their use of them. Teachers also recognize that using definitions in geometry is a means to an end rather than the end itself, which is the use of definitions to draw conclusions about space. Accomplished teachers, for example, might use definitions to prove theorems about parallel lines, congruent figures, similar figures, or circles.

Accomplished teachers know that the presence of proofs is not an isolated incident limited to geometry but is an integral part of mathematics as a whole, as well as of many non-mathematical domains. Teachers know that the basic isometric transformations—reflections, rotations, and translations—underlie the concept of congruence and that the concept of dilation, together with congruence, provides a precise definition of similarity. They also know that similar triangles are basic to understanding why the graph of a linear equation in two variables is a line.

Accomplished teachers know that an axiomatic system is set up solely for the purpose of organizing systematically a body of knowledge and that a valid presentation of geometry is not dependent on having a collection of axioms. At the same time, they are also aware that an axiomatic treatment of geometry leads to a fuller understanding of non-Euclidean geometries. They know that the most important aspect of any proof is to be explicit at each step about the hypothesis assumed and the conclusions to be proved.

Accomplished teachers know the basic geometric constructions and are aware that the solutions to some of the classical construction problems involving a straight edge and a compass depend on algebra and higher mathematics.

Accomplished teachers know that the essence of geometric measurement—be it length, area, or volume—is fixing a unit of measurement and then using the unit to “measure” another geometric figure, in the sense of “fitting” as many units or

fractional parts thereof as possible into the figure. They know that the most common units chosen for length, area, and volume are, respectively, the unit interval $[0, 1]$, the unit square, and the unit cube. Teachers understand the fundamental role played by the concept of congruence in geometric measurements.

Accomplished teachers can derive with ease the standard area formulas for rectangles, triangles, parallelograms, trapezoids, and circles, as well as the area of any polygon. Teachers can explain the volume formula of a cylinder with an arbitrary base, such as a circle, triangle, or regular polygon, and how it relates to the volume of the cone on the same base.

Trigonometry

Accomplished teachers know that the definitions of trigonometric functions for acute angles make sense only because of the theory of similar triangles. Teachers can use the unit circle to extend the definitions of the trigonometric functions from acute angles to all angles. They know that the x and y coordinates on the unit circle are the cosine and sine of the corresponding central angle. They also know that, at this point, the use of radians for angle measurement is more appropriate because the formulas in calculus are simplified by the use of radians. They know that the Pythagorean Theorem and the sine and cosine addition formula are all that is needed in trigonometric identities. Accomplished teachers of adolescents and young adults also know the relation of the sine and cosine addition formulas with the complex exponential function ($e^{i\theta} = \cos\theta + i\sin\theta$).

Discrete Mathematics

Discrete mathematics at the pre-university level is a non-unified collection of topics dealing with finite and discrete phenomena that are often connected to other areas of mathematics. Accomplished teachers are acquainted with certain topics about finite collections of objects, or infinite collections without involving limits. They know finite probabilities in terms of careful counting leading to standard permutations and combinations using binomial coefficients, such as how many ways can you make a group of three objects from a collection of five objects. They also know modular systems, for example clock arithmetic, and their arithmetic operations, including the fact that the arithmetic of base 2 is used in computers and coding.

Accomplished teachers know how to use truth tables to clarify the idea of logical implications, including the concepts of converse and contrapositive. For example, all teachers should know that the statement, “If it rains, then I bring an umbrella,” is not equivalent to the statement, “If I have an umbrella, then it is raining.”

An accomplished teacher is adept at handling infinite sequences, such as arithmetic and geometric sequences. They also know how to define sequences recursively, such as the Fibonacci sequence.

In addition, teachers of adolescents and young adults should be well acquainted with the basic definitions of graphs, including circuits, Euler circuits, and trees.

Data Analysis and Statistics

Accomplished teachers use both quantitative and qualitative approaches when answering questions involving data. To do so, they collect, organize, represent, and reason about data using a variety of numeric, graphic, and algebraic concepts and procedures, and they look for ways to describe and model patterns in data. They know how to interpret and draw inferences from data to make decisions in a wide range of applied problems, as well as how to use simulations to investigate situations.

Accomplished teachers understand that what separates the study of statistics from other areas of mathematics is the inherent variability in data. They understand the various sources of variability and how variability is at the heart of statistical reasoning and measurement. They understand why mean, median, and mode are useful, and they appreciate the meaning of each measure of variability within a context. Teachers understand that a goal of data interpretation is to help students become more discriminating consumers of information.

Accomplished teachers of adolescents and young adults understand the concepts behind basic inferential techniques and concepts such as confidence intervals and hypothesis testing. In addition, they are aware of the advantages, limitations, and appropriateness of each technique. They understand that statistical inference goes beyond describing data and involves using formal probabilistic methods to support or refute generalizations about populations based on samples, using the methods and language of probability.

Calculus

Accomplished teachers should be knowledgeable about the basic concepts of calculus, for example continuity, differentiation, and integration. For instance, teachers understand why there is a relationship between the limit of the function $1/x$ as x approaches zero and the y -axis as an asymptote of the function. They also understand the concept that the derivative of a function at a point can vary as the point changes. Whereas in algebra one can deal only with average rates of change, the mathematics teacher knows that with calculus one can address instantaneous rates of change. The teacher knows that because functions arise from all branches of science, calculus continues to play a central role in the sciences.

Accomplished teachers of adolescents and young adults are skilled in computations involving limits, derivatives, and integrals and their applications. For example, they can compute the area between two curves and two vertical lines. They are knowledgeable about the theoretical foundations of calculus so that they can help their students make sense of various computational procedures for problem solving. Teachers of adolescents and young adults understand the fundamental role of the limit concept in all of calculus (continuity, summation of infinite series,

differentiation, integration) and that the existence of these limits depends on the completeness of real numbers. They can supply reasoning for various theorems, such as the intermediate value theorem and the existence of maxima and minima for continuous function on finite closed intervals. They can also make use of technology to help students understand the limit process.

Standard III

Knowledge of Students

Accomplished teachers use their knowledge of human development and individual students to guide their planning and instructional decisions. They understand the impact of prior mathematical knowledge, home life, cultural background, individual learning differences, student attitudes and aspirations, and community expectations and values on students and their mathematics learning.

Accomplished teachers know their learners and use that knowledge to determine instruction. Accomplished teachers must know students well, both as early adolescent through young adult learners and as individuals learning mathematics. Mathematics teachers know that adolescents experience many intense emotional, physical, social, and intellectual changes over a relatively short period of time and that these changes may affect instruction and learning. At the same time, teachers know that the energy adolescents bring to the classroom might contribute to rich learning opportunities if channeled into appropriate activities. Teachers allow students to communicate their ideas while still guiding conversations toward the concepts being learned. Teachers also find ways to motivate students through connections to students' worlds. For example, teachers might relate mathematic problems to currently available technology. Teachers realize that beyond interests in material items, students are dealing with finding a sense of belonging, which can directly affect what they are willing to do in front of their peers.

Accomplished teachers recognize the variability in student development as students mature. Teachers understand the process of cognitive development in young people and know that students' cognitive profiles differ. Teachers work collaboratively with specialists, as necessary, and support every student while maintaining high standards. Teachers also recognize the wide-ranging mathematics backgrounds students have. Teachers help students internalize the language of mathematics and its processes while recognizing that students learn through varying approaches.

Accomplished teachers are also aware of the dispositions that students bring to and develop in the classroom. Such attitudes may include math anxiety, fear of failure, confidence in doing mathematics, perseverance, and valuing mathematics. These teachers are able to construct lessons and activities that build on and foster positive attitudes and minimize negative ones.

Accomplished teachers are aware of the different ways in which students learn mathematics. In designing lessons, accomplished mathematics teachers are sensitive to how students with differing strengths, interests, and ways of learning come to understand mathematics, and how these students develop the reasoning processes and attitudes that characterize mathematical thinking. Teachers continuously update their knowledge, staying abreast of changes and strategies that most effectively address the ever-changing needs of students. This may be accomplished by participating in lesson studies, attending conferences, and reading professional journals. For example, after attending a conference, teachers could enter into an action research study implementing ideas learned at the conference. Teachers recognize the merits and limitations of different approaches to teaching and realize that all students benefit from a multiplicity of approaches that allow them to consider important mathematical ideas and concepts from several perspectives. For instance, a teacher could transform real world data into a graphical representation and then into a symbolic form to help students internalize the concept of an exponential function. On the other hand, teachers recognize that all students need to increase their comfort level with abstract reasoning as they progress through grade levels. For example, the concepts of limits and infinity could be illustrated by examining how the limit of $1/2, 2/3, 3/4, \dots, n/(n+1)$... is 1.

The practice of accomplished teachers is distinguished by their capacity to integrate the goals of the curriculum with each student's knowledge base. Teachers notice those students in all groups who have developed exceptionally high abilities or affinities and tailor programs to provide challenges and opportunities that support these students. These teachers' lessons succeed, in part, because of their ability to recognize students' strengths and to assess, anticipate, and address student difficulties, understandings, and misconceptions. (See [Standard VII—Assessment](#).)

Accomplished teachers identify the strengths, interests, and experiences particular students bring to the mathematics classroom. When students do not have the prerequisite skills or have not had experiences conducive for studying a certain concept or skill, mathematics teachers adapt their teaching to acknowledge the skills and experiences of those students. Other strategies might include working with individual students, coordinating remediation opportunities, working with parents, or communicating with teachers in previous grades or courses. In addition, teachers know how to build upon students' strengths as they develop, using them to deepen students' knowledge in mathematics and to encourage them to apply mathematical understanding to other fields.

Accomplished teachers blend their knowledge of students, how students see mathematics, and how students develop new mathematical understandings into their instructional planning. These insights, along with the ability to identify exceptionalities in students, enable teachers to adapt their practice.

Standard IV

Knowledge of the Practice of Teaching

Accomplished mathematics teachers use their knowledge of pedagogy along with their knowledge of mathematics and student learning to inform curricular decisions; select, design, and develop instructional strategies and assessment plans; and choose materials and resources for mathematics instruction. Accomplished mathematics teachers stimulate and facilitate student learning by using a wide range of practices.

Accomplished teachers recognize that, while much of teaching is an art, there is a body of research that offers insight into teaching practices that can accelerate student learning in multiple ways. Teachers recognize that teaching affects student performance and is often the major effect on performance.

The repertoire of an accomplished teacher includes teaching strategies and best practices that engage students in exploring, discovering, and using mathematical ideas. Teachers design their lessons with well-defined mathematical goals in mind. Keeping in mind the prior mathematics experiences of students and the preparation necessary for future mathematical courses, teachers identify and make explicit the learning goals for each lesson. They are able to pace and sequence their objectives in order to meet their goals for students' learning. They articulate the objectives clearly and select instructional techniques and activities that enable students to meet them. Teachers know students, understand the concepts and procedures of mathematics, and know that various subject areas can enhance one another. They are adept at connecting mathematics to other disciplines. For example, mathematics teachers might help their students who are studying biology to see that the connections between how the amount of medicine one takes and how much is left in one's body are explained using exponential decay in mathematics. Or, teachers might apply the concept of the derivative to maximization of profit in economics. They know the curriculum standards and frameworks that connect mathematical ideas across grade levels. Teachers' choices are governed by their immediate and long-term instructional goals, the progress and interests of their students, the instructional opportunities that present themselves, and the particular dynamics of the teachers' classrooms.

Only by combining their knowledge of mathematics with what they know about students can accomplished teachers make well-crafted decisions as to what to

communicate and how to communicate mathematics to students. Teachers are versatile, expert communicators who can adapt both their style of presentation of the mathematics and their choices of what and how they teach.

Accomplished teachers take into account the individual needs and developmental levels of students when designing instruction. Teachers know how to observe and listen to students' interactions in order to blend instructional goals for the lesson with the learning goals of students. Teachers constantly reflect on the interaction between the purpose of the lesson and the requirements of the student to effectively satisfy both. Teachers use these observations to differentiate their instruction by providing different entry points to the same assignment with different skill sets. (See [Standard III—Knowledge of Students](#).)

Accomplished teachers motivate students based on their needs, interests, and intrinsic motivation. Teachers know the challenges and difficulties that students commonly encounter in learning particular mathematical topics, anticipate underlying misconceptions, and incorporate this prior knowledge in planning for instruction. For example, teachers understand the complexities of early number concepts and can trace conceptual difficulties of young adolescents back to misconceptions of elementary mathematics and make appropriate instructional decisions and accommodations to correct these errors. Mathematics teachers analyze student work to identify current mathematical understandings and use that information to drive instruction. For instance, mathematics teachers know a common misconception in the learning of algebra is that students often interpret the equal sign as an indicator that they should compute and find an answer, not as a statement of relationship that two things are the same. While preparing their unit on solving linear equations, teachers might use their knowledge of the misconception to plan their assessment of students' understanding of the meaning of equality.

Accomplished teachers may work actively within the school community to advance knowledge about the learning opportunities afforded by technology. Teachers help students learn about learning mathematics. The visual, computational, and interactive power of modern technology can be used to influence both the content and the methodology of student learning and assessment in mathematics. Teachers recognize opportunities afforded by technological tools—access to new ideas and new ways of representing and manipulating them—and effectively use the tools to deepen and enrich students' mathematical learning. Teachers encourage students to use technology to access information and communicate mathematically with others. When possible, teachers use appropriate technological tools to allow students to expand their mathematical tool kit. For instance, students might appreciate the symbolic notation if they accessed the history of its evolution easily on the Web. Teachers may select resources that simulate real situations in order to address problems that might be otherwise inaccessible. For example, in teaching experimental versus theoretical probability, teachers might use a calculator's or computer software program's random number generator to simulate large numbers of trials of flipping a coin. Teachers continually improve their own skills and fluency with technology and reexamine their teaching practices in light of what is possible.

Where access to technology is limited or nonexistent, accomplished teachers seek ways to acquire it. (See [Standard IX—Families and Communities](#).)

Accomplished teachers modify classroom plans and activities in response to student needs, interests, and unexpected opportunities for learning. Teachers demonstrate flexibility, insight, and responsiveness in dealing with the flow of the classroom. They recognize and respond to the mathematical potential of student questions and comments and pursue ideas of interest that emerge during classroom discussion. Teachers also help students reflect on and extend their learning, and they expect and help them to take responsibility for their learning. For example, teachers might provide opportunities for students to write in journals about what they have learned and what they need to focus on as a unit progresses. Because teachers understand different types of representational models, appropriate instruction and computational tools, and the strengths and weaknesses of each, teachers can select those best suited for different students and for different teaching situations.

Accomplished teachers know that classroom interactions can develop a life of their own, that no plan should be followed simply for its own sake, and that teachers must adapt their plans when appropriate. Teachers are prepared to adjust instruction—either because unforeseen difficulties suggest that a path they had planned to take will not succeed, or because a classroom discussion points to a beneficial alternative. For example, in learning linear functions and the slope as the rate of change, students might ask about what happens when the rate of change is not constant. While not planned, the teacher might decide to introduce examples of nonlinear functions to give students opportunities to compare and contrast the characteristics of linear and nonlinear functions. Teachers are able to anticipate misunderstandings and provide instruction that will help as ideas unfold. Furthermore, they choose topics for discussion wisely, relying on their understanding of what is appropriate and important.

Accomplished teachers foster learning by choosing imaginative examples, problems, and situations designed to interest and motivate students, illuminate important ideas, and support continued growth of student understanding. Teachers also know the importance of well-structured, scaffolded sets of problems that can be used to increase student understanding. Teachers work with small groups of students, asking clarifying or leading questions when necessary. Teachers involve students in decisions about mathematical topics or ways to study those topics. Mathematics teachers provide students with opportunities to reflect on their learning, and teachers serve as catalysts in launching student investigations. For instance, a teacher might show the application of the quadratic formula to prepare students to write an explicit representation of Fibonacci numbers.

Accomplished teachers promote meaningful discourse through the well-conceived questions they pose and through the rich tasks they provide. They demonstrate their use of appropriate questioning strategies by knowing how, when, and why to question students about their understanding of mathematics and provide a safe arena in which students can counter the arguments of others. Teachers encourage

students to pursue learning on their own. Teachers also know that a number of studies support the tremendous potential that discourse-intensive instruction has on student learning. Teachers use techniques that encourage discussion, such as restating students' observations, having students repeat and listen to one another, asking students to contribute to the discussion, and using wait time both before and after students respond. During discourse, teachers are aware of their own intonations and adjust their communication styles to the needs of the students and the context of the discussion. Teachers understand and are able to demonstrate strategic methods to communicate in a specific manner as they share information with students.

Accomplished teachers recognize that to teach mathematical vocabulary they must use it in the context of the content as well as in the context of the learner. For example, to meet the needs of English language learners, students who are auditorily challenged, or students with attention deficits, teachers need to use carefully chosen nonverbal cues, gestures, or color-coded symbols or cues to illustrate mathematics lessons. For instance, a polynomial with like terms may be presented for students to combine the terms. As students choose terms that may be added to one other, the teacher might represent like terms with different colors and shapes to help students create a simplified expression.

Accomplished teachers value mathematics. They take joy in it. They appreciate how knowledge in mathematics is created. They are excited by the ideas they explore with students. Teachers communicate that joy to their students. Teachers use the power of mathematics to fascinate students. Teachers elicit mathematical excitement and provide students with opportunities to experience the intellectual satisfaction that comes from finding a solution to a problem or justifying a conjecture. Instead of simply telling students how to solve a problem, the teacher may scaffold activities to help students recognize and celebrate their ability to use their knowledge of mathematics to answer questions.

Accomplished teachers epitomize the character they want to instill in students. Teachers demonstrate curiosity, respect, patience, honesty, fairness, and commitment to their art. They focus on and support students in their aspirations for high performance in mathematics and life-long learning. Teachers create a safe climate conducive to student learning and advocate for students. Teachers must quickly address instances of cheating, plagiarism, bullying, and harassment. They are exemplars of integrity who are open and approachable to students and the community. Teachers also recognize the essential importance of providing high-level engagement, critical-thinking activities, and authentic challenges for students. Mathematics teachers engage in principled practice, balancing multiple demands and goals to ensure that all students have optimal opportunities to learn.

The Teaching of Mathematics

The ways a teacher makes decisions and implements plans in the classroom provide the most visible and, arguably, the most important demonstrations of accomplished practice. The next three standards describe the types of tasks teachers construct and select, the ways teachers facilitate classroom discourse, and the practices teachers use to assess and monitor learning. Accomplished mathematics teachers successfully perform these functions through the roles they assume, the organizational schemes they use, the decisions they make, and the ways they adjust their plans from moment to moment.

Standard V Learning Environment

Accomplished mathematics teachers create environments in which students are active learners, show willingness to take intellectual risks, develop self-confidence, and value mathematics. This environment fosters student learning of mathematics.

Accomplished teachers use their knowledge of how students learn to create a stimulating and productive environment in which students are empowered to do mathematics. Teachers foster a respectful, engaging, and cooperative atmosphere for learning. They help students learn about learning mathematics. From the beginning of the school year, teachers engage their students in creating a community of learners in which students value taking intellectual risks.

In such an environment of trust, students feel safe to communicate different points of view, to conduct open-ended explorations, to make mistakes, and to admit confusion or uncertainty in order to learn. For example, before classroom discussions, the teacher might articulate norms designed to establish trust. In a middle school classroom, the teacher might create these norms, whereas in a high school classroom, the students could help develop the norms. In these classrooms, students develop a strong work ethic and assume ownership and responsibility for their learning, so that students along the learning spectrum benefit. When one student develops an understanding of a concept, he or she uses this new knowledge to help other students understand that concept. Creating and maintaining such a learning

environment requires skill and planning, a variety of instructional methods, flexibility, good judgment, and discretion.

Accomplished teachers consider the mathematical understandings, needs, interests, and working styles of their students and the mathematics they are studying. Teachers recognize the multiplicity of challenges and continually seek ways to help students thrive. Teachers create a culture in which each student learns to value mathematics and experiences success in doing mathematics. Teachers lead by example and convey to students the delight that comes with the command of a mathematical tool or principle. Teachers help students develop the ability to work both independently and collaboratively on mathematics, recognizing that the long-range goal of a teacher is to help students become self-directed and capable of learning on their own.

An accomplished teacher constantly reflects on ways to improve the learning environment. Teachers know students well and create productive learning environments through the use of classroom management strategies. Teachers know what motivates, interests, and inspires students, as well as what frustrates them. Mathematics teachers can establish classroom routines and policies that allow students to focus on learning. For example, teachers establish protocols during cooperative learning activities that revolve around encouraging discussion about mathematics or whole-class discussion norms that help student-led discussions focus on evidence and support discourse. Regardless of how mathematically rich the learning environment is, challenges still exist in helping students to learn because of the many aspects of students' lives that they bring to the classroom. (See [Standard III—*Knowledge of Students*](#).)

The look of the classroom of an accomplished teacher also tells something about the role mathematics plays in students' everyday lives in school. Student work, mathematical models, and manipulative materials likely to pique students' interests and encourage their involvement in mathematics are evident in these teachers' classrooms. The physical arrangement of space and furniture, along with teachers' use of space is purposeful and designed to foster mathematical discourse and support both collaborative and independent student work. Teachers working in circumstances in which they have little or no control over their physical setting make whatever accommodations they can to contribute to students' learning in and thinking about mathematics.

In addition to creating a mathematically rich learning environment, teachers, when possible, create a technology-rich classroom. For instance, there are many Web sites and applets that are interactive and that illustrate mathematics topics such as transformations, isometric views of 3-D shapes, tangent lines, limits, and areas under a curve. With the prevalence of technology in students' worlds outside the classroom, teachers strive to use technology geared toward engaging students in the learning of mathematics.

Standard VI

Ways of Thinking Mathematically

Accomplished mathematics teachers develop their own and their students' abilities to reason and think mathematically—to investigate and explore patterns, to discover structures and establish mathematical relationships, to formulate and solve problems, to justify and communicate conclusions, and to question and extend those conclusions.

Accomplished teachers bring insight about mathematics to students, including new perspectives on standard problems and unexpected connections among different fields. Teachers are proficient not only in solving problems, but also in making students aware of different strategies for solving a problem, as well as the relative merits of each. They have the confidence to help students face uncertainties and make strategic decisions in exploring unknown territories.

Accomplished teachers know that mathematics is a discipline of concepts, principles, procedures, and reasoning processes. Thinking mathematically includes representing, modeling, proving, experimenting, conjecturing, classifying, visualizing, and computing. In the classrooms of accomplished teachers, students are engaged in identifying patterns; solving problems; reasoning; forming and testing conjectures, justification and proof; and communicating results. Students search for connections and solve problems, while reflecting on both the mathematics and their own thought processes.

Accomplished teachers recognize that important general concepts and reasoning methods undergird the development of mathematical power. They model mathematical reasoning as they work with students and encourage students to question processes and challenge the validity of particular approaches. Students make conjectures and justify or refute them, formulate convincing arguments, and draw logical conclusions. Sound reasoning—not an edict from the teacher—is the arbiter of mathematical correctness. In short, students become mathematically empowered as they learn to think, reason, and communicate mathematically.

Accomplished teachers recognize that mastering mathematical facts and procedures is only a part of what it means to learn mathematics. Teachers must understand and consistently employ mathematical thinking processes in their classroom practice that include the following:

- Reasoning correctly using processes such as classification, representation, deduction and induction;
- Using heuristics as a key strategy to guide solutions to mathematical problems, such as testing extreme cases, conducting an organized search of specific examples, and using different problem representations;
- Modeling mathematical relations in problem situations—describing important relationships through symbolic expressions and other representations;
- Connecting ideas, concepts, and representations across the strands of mathematics.

Teachers also know the importance of developing students' understanding of and disposition to do mathematics. Teachers realize that teaching students to “think mathematically” means helping them develop a mathematical point of view in which they consistently use the mathematical thinking processes listed above; recognize situations in which mathematical reasoning might be useful; and have the ability, skill, and confidence to think through a mathematical situation. For example, the teacher might stress the idea that knowing how to calculate the area of a triangle is all that is needed for knowing how to calculate the area of all polygons. To encourage mathematical thinking, teachers provide settings that allow students to test mathematical ideas, patterns, and conjectures; discover principles; synthesize evidence; and apply their growing knowledge to a variety of problems. Teachers know and use the overarching themes of mathematics that help students understand and appreciate the powerful relationships between mathematical ideas and problems—as in making students aware of the relationship between diverse fields, such as algebra and geometry or geometry and probability. For instance, teachers might discuss how similar triangles are basic to the understanding of linear functions and how the concept of area makes tangible the concept of probability.

Accomplished teachers know multiple ways to represent mathematical concepts, and they organize tasks so that students will learn that a single problem may have many representations. Teachers encourage students to distinguish between these representations and to select a compelling and efficient representation for a given problem or situation. Teachers know the importance of developing mathematical concepts concretely, so they are knowledgeable about the use of a variety of representations that support their instructional goals. For instance, teachers might use a dynamic geometry software package to help students develop definitions for and characteristics of plane figures using a graphic organizer. Teachers know and communicate that representations are often needed to form abstractions initially, and they allow students to think mathematically about abstract concepts. Although not every teacher will use the same materials for the same purposes, mathematics teachers are adept at using concrete materials that help students develop various mathematical understandings. For example, students might use concrete materials or manipulatives to develop a rule for binary operations with integers. Teachers help students make connections between their manipulative experiences and the mathematical ideas they need to grasp.

Accomplished teachers provide students with problems and applications that will allow them to explore new mathematical content, reflect on the problem-solving process, extend and refine their thinking, make generalizations about the procedures they have used, and link those generalizations with what they have learned previously. Teachers provide many rich opportunities for students to apply mathematics to interesting problems. In so doing, teachers point out the interrelated domains of mathematics. They not only choose tasks related to everyday life—including the sciences, economics, politics, or business—but they also choose tasks that will extend understanding within mathematics. Their choice of problem contexts reflects the breadth of mathematics and its applications.

Accomplished teachers deliberately structure opportunities for students to use and develop appropriate mathematical discourse as they reason and solve problems. These teachers give students opportunities to talk with one another, work together in solving problems, and use both written and oral discourse to describe and discuss their mathematical thinking and understanding. As students talk and write about mathematics—as they explain their thinking—they deepen their mathematical understanding in powerful ways that can enhance their ability to use the strategies and thought processes gained through the study of mathematics to deal with life issues. For example, when students in geometry are given six toothpicks and asked to construct with them four equilateral triangles, they usually conclude after multiple attempts that it is impossible. When probed to explain their thinking, they focus on the mathematics they know and usually conclude that this cannot be done on the plane, which is true. Then they realize that they were not asked to do it on the plane and immediately come up with the solution: the regular tetrahedron. From this exercise, they learn to think through the analysis of a problem or situation in any discipline. Mathematics teachers encourage students to confront and challenge ideas and to question peers as they discuss mathematical ideas, develop mathematical understanding, and solve mathematical problems. Teachers use probing and supportive questions to advance students' thinking. Teachers monitor what students do, using mathematical communication regularly to help students build understanding. For example, teachers might use the graphing of rational functions to help students understand the concept of an asymptote.

Accomplished teachers are well aware that students' mathematical achievement is still mainly dependent on their ability to conceptualize and analyze mathematics, to discover structures and establish relationships, to explore justification and proof, and to formulate and solve problems. For this reason, teachers know that they must develop students' mental acuity as well as pencil-and-paper skills. In addition, the latest technology has provided effective tools to help develop students' reasoning, mathematical thinking, and discourse. Accomplished teachers are able to use applications such as graphing technology, interactive geometry software, and computer algebra systems to support student inquiry, conjecture, and proof. For instance, when using geometric software, students can explore the properties of a parallelogram and conjecture about the consecutive angles being supplementary. This could lead students to "see" the proof as to why this is true. Teachers also know how to use calculators as exploratory tools to develop students' understanding of mathematics.

Standard VII Assessment

Accomplished mathematics teachers integrate a range of assessment methods into their instruction to promote the learning of all students by designing, selecting, and ethically employing assessments that align with educational goals. They provide opportunities for students to reflect on their strengths and weaknesses in order to revise, support, and extend their individual performance.

Accomplished teachers continuously gather available achievement data about their students in order to adjust their instruction. Teachers use formal and informal assessments to provide a holistic view of a student's mastery or need for further instruction. Teachers assess students' understanding of mathematical concepts and procedures, their fluency at operations, and their ability to effectively use resources. Teachers also assess the students' clarity of communication, their ability to innovate and apply multiple strategies to problems, and their ability to deal with topics they are learning. For example, teachers might regularly provide opportunities for students to write a reflection or justification. Based on what the students write, teachers can address misconceptions, lack of clarity or completeness, and level of understanding on any given topic.

Accomplished teachers view ongoing assessment as an integral part of their instruction, benefitting both the teacher and the student. Teachers—guided by well-defined instructional goals of the current class and students' future mathematical classes—design appropriate assessments and activities to monitor the progress of the class as a whole, as well as the work of individual students. Mathematics teachers skillfully incorporate opportunities for assessing students' progress into daily instruction. Using levels and types of questioning, teachers assess students' abilities to comprehend, apply, and synthesize. Teachers use a variety of strategies to explore and expand their students' thinking and a variety of methods to check for students' understanding, and teachers use this information to shape their teaching. Throughout the assessment process, teachers monitor the skills that students may be missing and find ways to design or modify instruction to minimize gaps in learning.

Accomplished teachers help students develop the ability to self-monitor and evaluate personal progress. For students to become self-reliant learners, teachers know that students must assume increasing responsibility for their own learning at an appropriate developmental level. Teachers encourage students to set high goals,

both for the current class and in preparation for future mathematics courses, and teachers show students how to evaluate their progress toward those goals. Teachers engage their students in learning from other students' work. In this way, students gain a fresh perspective on their own work, as well as opportunities to reflect on and evaluate their progress and to revise, support, and extend their learning. Teachers set high expectations and ensure that those expectations are clear and understood by all their students. Mathematics teachers establish clear criteria for success. Students know what to expect when they are assessed and regularly receive opportunities for open-ended reflection about a task, a unit, an experience, or their learning in general. For example, to keep track of how well students convey their mathematical ideas using appropriately precise terms and well-organized reasoning, teachers may keep a discourse log that focuses on clarity of communication during students' presentations of their work.

Accomplished teachers use formative assessment results to modify their lessons and learning opportunities and activities. Accomplished teachers recognize that the form of assessment is not nearly as important as how well it relates to classroom instruction. Teachers use assessment strategies to identify student strengths and areas for improvement, and they provide timely and constructive feedback. They use assessment to communicate and demonstrate that students are learning mathematics. The students of accomplished teachers come to value assessment as an important means of evaluating their own progress and may not distinguish assessment from instruction. For example, teachers may incorporate into lessons one or two problems that often elicit particular misconceptions about the topic of instruction and observe students' work on these problems to assess whether students may have developed the misconception.

Besides assessing student learning using teacher-designed assessments, accomplished teachers are keenly aware of any assessment that originates outside the teacher's classroom that plays an important part in students' educational experiences and know the purposes and content of these assessments. In such external testing situations, mathematics teachers do not abandon their instructional goals; rather, they incorporate the pertinent mathematics objectives of the test into their overall instructional plan. For instance, instead of self-contained test preparation for state assessments, teachers may plan their units and lessons to incorporate objectives at appropriate points in the development of concepts. Teachers may also incorporate some items in classroom assessments that are similar in format to those of the state assessment. They evaluate their own curricular decisions in light of the content of these tests, ensuring that their students are well prepared for those examinations that will be important to their future goals. Teachers read and consider the curricular implications of data from local, state, national, and international comparisons. Teachers recognize the importance and role of data, especially longitudinal data, to inform instruction and curricular choices.

Teachers have an ethical obligation to clearly communicate what the learning goals and grading expectations are and to accurately report how well students have achieved those goals and met those expectations.

Professional Development and Outreach

Accomplished mathematics teachers, according to the Five Core Propositions, think systematically about their practice and learn from their experiences. As professionals, they reflect on their teaching and periodically adjust their course content and instruction to respond to new ideas and the rate of progress they observe in their students.

Teachers must show persistence and creativity in their efforts to involve families in meaningful ways. Today, more than ever, teachers must reach beyond their classrooms. They must participate in broader learning communities and reach out to the community at large. At the school, district, state, and national levels, teacher involvement is a key component in the effort to improve educational opportunities and learning for all students. The definition of who and what a teacher is continues to evolve toward an emerging picture of a talented and competent professional who is charged with the most important function in society—educating tomorrow’s adults. Important aspects of this professionalism are reflected in the final cluster of standards.

Standard VIII Reflection and Growth

To improve practice, accomplished mathematics teachers regularly reflect on what they teach, how they teach, and how their teaching impacts student learning. They keep abreast of changes and learn new mathematics and mathematical pedagogy, continually improving their knowledge and practice.

For accomplished mathematics teachers, every class and every course provide the opportunity to reflect and improve on an ongoing basis. Teachers modify their teaching practices based on their experiences and on the continuous process of self-examination, using a variety of strategies to collect data about their own teaching. They also gather information from students about the effectiveness of their teaching. For example, if a student is having difficulty with a particular aspect of algebra, a teacher might reexamine instructional methods and choices in order to help the student better understand the concept. This information may come from

formative and summative assessments, classroom observation, homework, student conferences, or student surveys and forms the basis for ongoing improvement in teachers' knowledge and practice.

The body of mathematics and the pedagogical bases for teaching mathematics are dynamic. The knowledge base of mathematics, mathematical theories and applications, and the evolution of technology, such as fractals, recursion, and computer Web design, present opportunities that change the way people engage in mathematical reasoning.

Accomplished teachers regularly engage in solving problems in which solutions are not obvious. They do this to increase their mathematical knowledge and to mimic the experiences of their students. Through this activity, teachers gain insight into how difficult it can be to persevere in a problem-solving situation. For example, a teacher might take a higher-level mathematics course or solve problems from mathematics textbooks beyond the level they teach, from science textbooks, or from contests.

Accomplished teachers keep abreast of strategies for improving mathematics learning and teaching through such activities as reading professional journals, dialoguing with peers, attending meetings and conferences, and participating in professional organizations. They continually participate in professional development and regularly refine their practices in light of professional knowledge and experience, while keeping in mind the intended effect on students when implementing any new knowledge such as research on the structures and development of the adolescent brain in relation to student decision-making. Whether extending their formal education or engaging in a self-directed plan, teachers actively pursue ways of enhancing their own knowledge and skills. They identify areas for self-improvement and seek strategies for reaching their educational and instructional goals.

Accomplished teachers work actively with colleagues to use their knowledge and understanding of mathematics and students to enhance student learning. Realizing the positive effect that shared experiences can bring to their own teaching practices, teachers may participate in lesson study or form professional learning communities. The idea of a professional learning community may include meeting with colleagues from within the content area, meeting with an interdisciplinary group of colleagues, or becoming an active member of an educational organization. These meetings may be face-to-face or via technology, informal or formal, regular or as needed. Teachers take from these discussions valuable information that will enhance their practice, while recognizing the cyclical nature of teaching and remaining open to examining their practice as ongoing professional development. At the core of reflection and professional development is the impact it will have on student learning.

Although accomplished teachers are alert to new developments in mathematics, technology, and mathematics teaching, they do not adopt blindly each new pedagogical method that becomes popular. Rather, they respect both new and time-tested thinking about mathematics teaching. For example, teachers understand that, although there is still a need for seeing geometry as a mathematical system with

definitions and structure, students also understand geometry from the viewpoint of a dynamic system where the representations can be translated and transformed in ways that illustrate concepts in powerful ways. Teachers filter ideas through the lens of their own experiences and the particular needs of their students, incorporating new ideas as they fit those needs.

Standard IX

Families and Communities

Accomplished mathematics teachers collaborate with families and communities to support student engagement in learning mathematics. They help various communities, within and outside the school building, understand the role of mathematics and mathematics instruction in today's world.

Accomplished teachers recognize the complexity of the home-school-student relationship, and they view families as partners who can encourage students to appreciate and value mathematics. Teachers respect the role of family members as students' first teachers and as supporters of students' growth and development. Teachers look to families for information about students' strengths, interests, dispositions, habits, and home life. Teachers actively seek to learn about the various cultures from which their students and their families come, respecting their values and recognizing that cultural differences may affect instruction.

Realizing that family members may or may not have had positive experiences in their own mathematical education, accomplished teachers create a non-threatening environment for family collaboration. Teachers find ways to communicate the value of mathematics, as well as its potential to open doors to students' futures. For example, the teacher may use newsletters, e-mail, a personal Web site, or family nights to illuminate the utility of mathematics. Connecting the value of mathematics to real world opportunities and challenges builds a rapport between teachers and family members who work together to ensure positive and productive mathematical learning experiences. Teachers work to build two-way communication with families and encourage family members to visit or communicate with the teacher whenever possible, with a focus on creating a positive learning environment for students.

Using information concerning the practical value of mathematics, accomplished teachers enter into conversations about the content and practice within the classroom and school. They share current research that informs their pedagogical decisions, such as cooperative groups and patterns of discourse. They communicate with families the content of the curriculum and invite family members to learn about that content in a non-threatening manner. For instance, a teacher may host a dessert exchange where family members learn more about what students are studying and doing in mathematics. To create an instructional collaboration, teachers find ways to engage the family in the mathematics that the student is learning.

When discussing students' education, accomplished teachers focus their discussions on both the positive and challenging aspects of students' knowledge and patterns of engagement. Recognizing that the home community plays an important role in the success of the students, teachers are sensitive to working with families.

In addition to the home community, accomplished teachers recognize the importance of the school community as it relates directly to student learning. Just as teachers build a collaborative environment between the home and school, they also build a collaborative environment between other educators within the school. Teachers work effectively with other school professionals to make students a part of a community with a coherent ethos. Accomplished teachers may work actively within the school community to advance knowledge about the learning opportunities afforded by technology. They help other teachers integrate knowledge across the disciplines in areas of applications, connections, and extensions. For example, they may work with language arts teachers to share ideas on communication skills and writing in mathematics, with social studies teachers on incorporating problem solving or data analysis, or with visual arts teachers on spatial relationships. In addition to building an interdisciplinary connection, teachers become ambassadors for mathematics. For example, they may share ideas for fair apportionment with the social science teachers, ideas for tessellating figures with art teachers, and ideas for technical mathematics writing with English teachers. They take on the responsibility of informing the school community, counselors, principals, and other teachers about the value of mathematics and the doors and opportunities mathematics opens for students. Recognizing that school professionals, much like some family members, may feel that they struggle with mathematics, mathematics teachers build awareness in their school community of the negative impact such language can have on a student or family member. Mathematics teachers work with counselors to promote the value of mathematics and the possibilities of furthering students' mathematical education or career.

Standard X

Professional Community

Accomplished mathematics teachers continually collaborate with other teachers and education professionals to strengthen the school's mathematics program, promote program quality and continuity across grade levels and courses, and improve knowledge and practice in the field of mathematics education.

Accomplished teachers are members of a professional mathematical community and work in a collaborative environment. Seeing themselves as partners with other teachers, they are dedicated to improving the profession. They care about the quality of teaching in their schools, and, to this end, their collaboration with colleagues is continuous and explicit. They recognize that collaborating in a professional learning community contributes to their own professional growth, as well as to the growth of their peers, for the benefit of student learning. Teachers promote the ideal that working collaboratively increases knowledge, reflection, and quality of practice and benefits the instructional program. Teachers seek to make their teaching an open community activity. They realize that while it may not be feasible to have a truly open classroom where colleagues may observe lessons followed by a collegial discussion, it may be possible to have an open classroom by sharing instructional anecdotes, sample products, or videotaped lessons. Teachers observe and study other teachers' practices, engage colleagues in dialogue about professional issues, and may serve as mentors to new teachers as well as coaches to experienced colleagues. Decision making becomes a collaborative process rather than an isolated process. Students become "our" students rather than "my" students in an effort to utilize all available resources. Teachers collaborate with parents, colleagues, and other members of the community in the education of early adolescents through young adults.

Because accomplished teachers are dedicated to their students and to their learning of mathematics, the focus of teachers' contributions to the larger learning community often revolves around curricular or assessment issues. These may include periodically reviewing district and school curricula, textbooks, external and internal testing instruments, and practices in their school or district to help ensure that materials and guidelines are thoughtfully organized; consistent with the profession's views of best practices; tailored to the students and the community they serve; and aligned with the goals of the mathematical community, school, and classroom.

As an extension of their local responsibilities, accomplished teachers engage in a wide range of professional activities. They contribute productively to the advancement of their field through active participation in professional organizations at

the local, state, regional, and national levels. They work to support the improvement of mathematics education and the professional growth of mathematics teachers at all levels including pre-kindergarten through 16 as well as preservice and in-service teachers. For example, teachers might partake in action research or partner with a university to participate in research projects. Such commitment is central to teachers' dedication to the quality of their practice and to the advancement of mathematics education.

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